

CHAPTER-4

**TRANSIENT HEAT
CONDUCTION**

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Whenever the boundary temperatures change the temperature at each point of the system change with time until steady-state temperature distribution is reached. These are categorized as ***unsteady***, or ***transient*** heat conduction. Cooling of a hot metal billet with air or water is a typical example.

The simplest situation is where temperature gradients within the solid are small such that uniform temperature can be assumed at any time. The analysis that uses this approach is termed as ***lumped capacitance method***. Following this, analytical and numerical methods will also be seen.

4.1 THE LUMPED CAPACITANCE METHOD

A typical application in heat treatment is quenching where a metal or an alloy with initial temperature T_i is suddenly immersed in a liquid of lower temperature, $T_\infty < T_i$. This is shown in [fig-chp5\fig5.1.pptx](#). If quenching begins at time $t = 0$, then given sufficient time, the temperature of the solid will decrease eventually to T_∞ . The heat transfer is due to convection on the surface. The assumption, here, is that the temperature of the solid is ***spatially uniform***. This assumption, according to Fourier's law implies infinite k which is clearly impossible. But when

compared to the convective heat transfer resistance, it can approximately satisfy the above condition.

Energy balance on the surface gives

$$\dot{E}_{\text{in}} + \dot{E}_{\text{g}} = \dot{E}_{\text{out}} + \dot{E}_{\text{st}} \quad \dot{E}_{\text{in}} = \dot{E}_{\text{g}} = 0$$

After substitution

$$-\dot{E}_{\text{out}} = \dot{E}_{\text{st}} \quad -hA_s(T - T_{\infty}) = \rho Vc \frac{dT}{dt}$$

Let $\theta \equiv T - T_{\infty}$, and this will give

$$\frac{\rho Vc}{hA_s} \frac{d\theta}{dt} = -\theta$$

Separating the variables and integration gives

$$\frac{\rho V c}{h A_s} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = - \int_0^t dt$$

where $\theta_i = T_i - T_{\infty}$ is the initial condition

Evaluating the integrals will give

$$\frac{\rho V c}{h A_s} \ln \frac{\theta}{\theta_i} = -t \quad \text{or}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp \left[- \left(\frac{h A_s}{\rho V c} \right) t \right]$$

The above equation gives the

- time required for the temperature to reach T or
- Temperature at a prescribed time t

The solution also indicates exponential decay as shown in [fig-chp5\fig5.2.pptx](#) .

If we define τ_t as the thermal time constant, which is an indicator of how fast the solid will respond to surrounding temperature change

$$\tau_t = \left(\frac{1}{hA_s} \right) (\rho V c) = R_t C_t$$

Where

R_t = convection heat transfer resistance

C_t = thermal capacitance (lumped)

Increase in τ_t (due to increase in R_t or C_t or both)

the solid will respond slowly.

Total heat transfer up to some time t will be

$$dQ = \dot{q}dt = hA_s(T - T_\infty)dt = hA_s\theta dt$$

$$Q = \int_0^t \dot{q}dt = hA_s \int_0^t \theta dt$$

Substitution of the solution gives

$$Q = (\rho Vc)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right) \right] = E_{\text{out}} = -\Delta E_{\text{st}}$$

For quenching Q is positive and the solid experiences a decrease in energy.

5.2 VALIDITY OF THE LUMPED CAPACITANCE METHOD

The previous method is the simplest and the most convenient method for the solution of transient problems. But it comes at a cost

- the assumption made-infinite thermal conductivity

Under what condition will this assumption hold?

Consider the steady-state condition shown in [fig-chp5\fig5.3.pptx](#) . $T_{\infty} < T_{s,2} < T_{s,1}$.

Under steady-state condition

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_{\infty})$$

Rearranging results in

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{L / kA}{1 / hA} = \frac{R_{t,cond}}{R_{t,conv}} = \frac{hL}{k} \equiv Bi$$

Bi-Biot number-a dimensionless parameter which is a ratio of the two thermal resistances.

$Bi \ll 1$: $R_{t,conv} \gg R_{t,cond}$ - equivalent to $k \rightarrow \infty$

$Bi \gg 1$: $R_{t,conv} \ll R_{t,cond}$ – not good for lumped capacitance approach

For transient processes, consider [fig-chp5\fig5.4.pptx](#). The block is initially at T_i and experiences cooling when it is immersed in a fluid of $T_{\infty} < T_i$.

For $Bi \ll 1$ the temperature gradient in the block is small and $T(x,t) \approx T(t)$.

For moderate to large values of Bi -temperature gradient is significant.

The lumped capacitance method will hold true for

$$Bi = \frac{hL_c}{k} < 0.1$$

Formula to be used for different geometries would require the definition of L_c , a ***characteristic length***, as

$$L_c = V/A_s$$

Plane wall: wall thickness L , $L_c = (LhW)/2hW = L/2$

Cylinder: radius r_o , $L_c = (\pi r_o^2 W)/2\pi r_o W = r_o/2$

Sphere: radius r_o , $[4/3(\pi r_o^3)]/4\pi r_o^2 = r_o/3$

For conservative approach, L_c values

Plane wall: same cylinder and sphere: r_o

Using $L_c = V/A_s$, the exponent of the transient equation may be expressed as (after some manipulation)

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k} \cdot \frac{k}{\rho c} \cdot \frac{t}{L_c^2} = \frac{hL_c}{k} \frac{\alpha t}{L_c^2}$$

The above is a product of two dimensionless numbers

$$\frac{hA_s t}{\rho V c} = \text{Bi.Fo} \quad \text{where } \text{Fo} \equiv \frac{\alpha t}{L^2}$$

Fo is called ***Fourier number***-dimensionless time.

Substitution in the solution gives

$$\frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp(-\text{Bi.Fo})$$

Example 5.1

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400 \text{ W/m}^2\cdot\text{K}$, and the junction thermophysical properties are $k = 20 \text{ W/m}\cdot\text{K}$, $c = 400 \text{ J/kg}\cdot\text{K}$, and $\rho = 8500 \text{ kg/m}^3$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C , how long will it take for the junction to reach 199°C ?

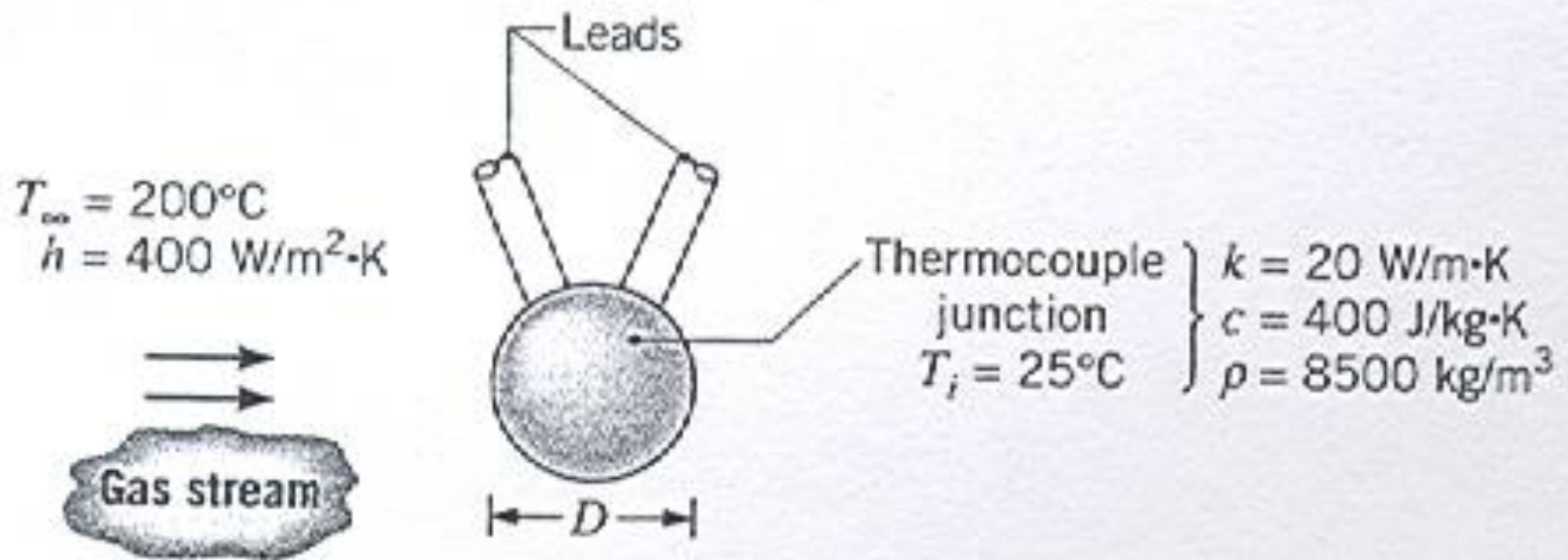


Figure for example 5.1

Solution

$$1. A_s = \pi D^2 \quad \text{and} \quad V = \pi D^3/6$$

$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho\pi D^3}{6} c$$

Substituting numerical values

$$D = \frac{6h\tau_t}{\rho c} = \frac{6 \times 400 \times 1}{8500 \times 400} = 7.06 \times 10^{-4} \text{ m}$$

With $L_c = r_o/3$

$$Bi = \frac{h(r_o/3)}{k} = \frac{400 \times 3.53 \times 10^{-4}}{3 \times 20} = 2.35 \times 10^{-3}$$

Lumped capacitance method is an excellent approxim.

2. The time required for the junction to reach 199°C is

$$t = \frac{\rho(\pi D^3 / 6)c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{8500 \times 7.06 \times 10^{-4} \times 400}{6 \times 400} \ln \frac{25 - 200}{199 - 200}$$

$$t = 5.2 \text{ s} \approx 5\tau_t$$

5.3 GENERAL LUMPED CAPACITANCE ANALYSIS

Transient heat conduction can also be initiated by radiation; by a heat flux from a sheet of electrical heater attached to a surface, etc.

[fig-chp5\fig5.5.pptx](#) depicts the influence of convection, radiation, an applied surface heat flux and internal energy generation.

Applying energy conservation principle

$$q_s'' A_{s,h} + \dot{E}_g - (q_{\text{conv}}'' + q_{\text{rad}}'') A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

$$q_s'' A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon \sigma (T^4 - T_{\text{sur}}^4)] A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

The above is a non-linear, first order, non-homogeneous differential equation which can not be integrated to obtain an exact solution
-requires approximate solution by numerical (finite difference) approach.

Exact solution for simplified equation:

(a) If no imposed heat flux or internal generation and convection is absent (vacuum) or negligible relative to radiation, the above equation simplifies to

$$\rho V c \frac{dT}{dt} = \varepsilon A_{s,r} \sigma (T^4 - T_{sur}^4)$$

Separating the variables and integrating

$$\frac{\varepsilon A_{s,r} \sigma}{\rho V c} \int_0^t dt = \int_{T_i}^T \frac{dT}{T_{\text{sur}}^4 - T^4} \quad \text{gives}$$

$$t = \frac{\rho V c}{4 \varepsilon A_{s,r} \sigma T_{\text{sur}}^3} \left\{ \ln \left| \frac{T_{\text{sur}} + T}{T_{\text{sur}} - T} \right| - \ln \left| \frac{T_{\text{sur}} + T_i}{T_{\text{sur}} - T_i} \right| + 2 \left[\tan^{-1} \left(\frac{T}{T_{\text{sur}}} \right) - \tan^{-1} \left(\frac{T_i}{T_{\text{sur}}} \right) \right] \right\}$$

The above will not give T explicitly and no solution for $T_{\text{sur}} = 0$ (deep space). However for $T_{\text{sur}} = 0$ in the above integral equation, it yields

$$t = \frac{\rho V c}{3 \varepsilon A_{s,r} \sigma} \left(\frac{1}{T^3} - \frac{1}{T_i^3} \right)$$

(b) For negligible radiation, and for $\theta = T - T_\infty$, $d\theta/dt = dT/dt$ the equation reduces to a linear, first-order, non-homogeneous differential equation

$$\frac{d\theta}{dt} + a\theta - b = 0 \quad a = \left(\frac{hA_{s,c}}{\rho V c} \right) \quad b = \left[\frac{\dot{q}'' A_{s,h} + \dot{E}_g}{\rho V c} \right]$$

The above can be converted to a homogeneous equation by using

$$\theta' \equiv \theta - \frac{b}{a} \quad \text{results in} \quad \frac{d\theta'}{dt} + a\theta' = 0$$

Separating variables and integrating from 0 to t (θ'_i to θ') gives the

solution as

$$\frac{\theta'}{\theta'_i} = \exp(-at) \qquad \frac{T - T_\infty - (b/a)}{T_i - T_\infty - (b/a)} = \exp(-at)$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty} [1 - \exp(-at)]$$

For steady state $t \rightarrow \infty$, the equation reduces to

$$(T - T_\infty) = (b/a)$$

Example 5.2

Consider the thermocouple and convection condition of example 5.1, but now allow for radiation exchange with the walls of a duct that encloses the gas stream. If the duct walls are at 400°C and the emissivity of the thermocouple bead is 0.9, calculate the steady-state temperature of the junction. Also, determine the time for the junction temperature to increase from initial condition of 25°C to a temperature that is within 1°C of its steady-state value.

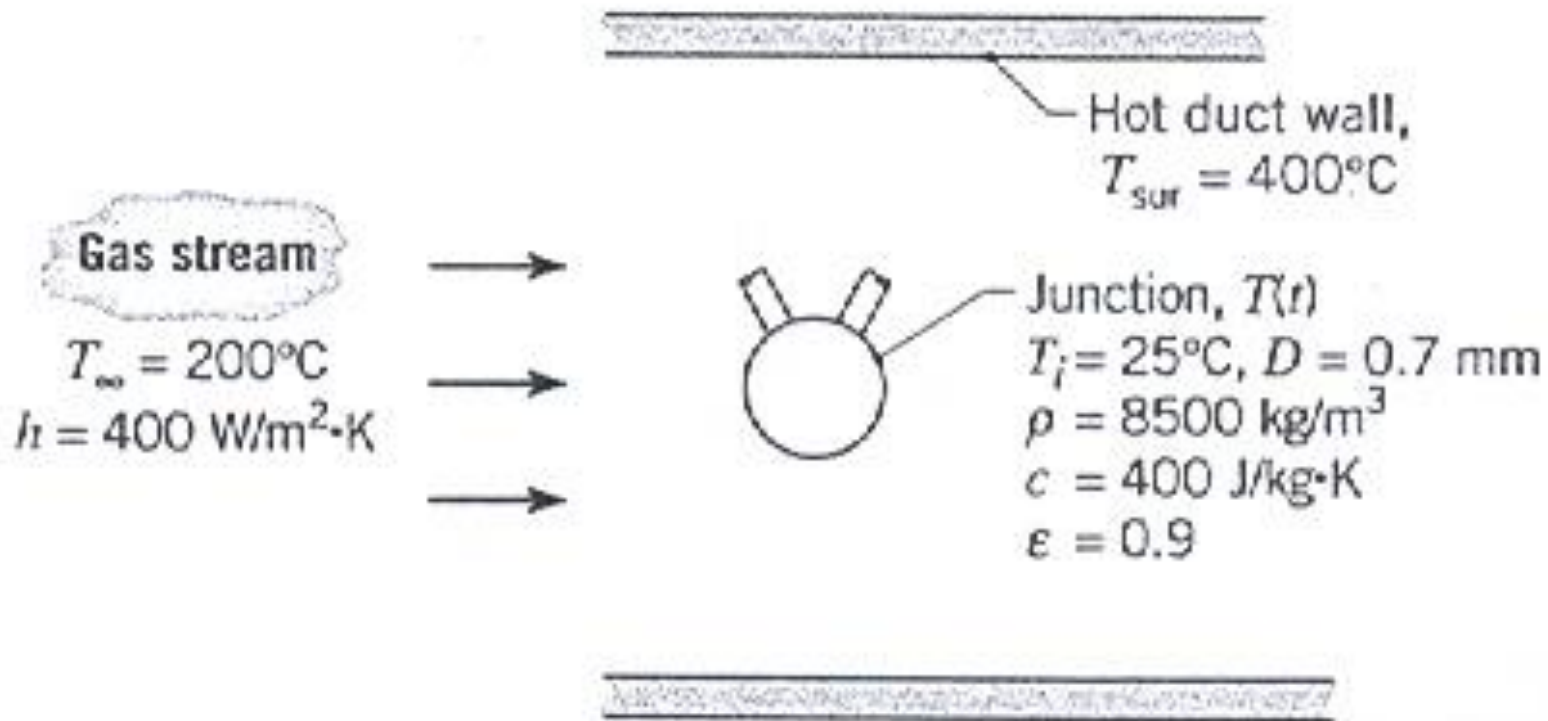


Figure for example 5.2

Solution

1. Energy balance on the thermocouple $\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$

$$[\varepsilon\sigma(T_{\text{sur}}^4 - T^4) - h(T - T_{\infty})]A_s = 0$$

Substituting numerical values gives

$$T = 218.7^{\circ}\text{C}$$

2. As this involves the transient part, the complete equation is $\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$

$$[-h(T - T_{\infty}) + \varepsilon\sigma(T_{\text{sur}}^4 - T^4)]A_s = \rho Vc \frac{dT}{dt}$$

Numerical solution for $T = 217.7^{\circ}\text{C}$ gives $t = 4.9 \text{ s}$

5.4 SPATIAL EFFECTS

For a one dimensional problem, the transient heat conduction equation can be determined from the general heat conduction equation as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

The solution will require two boundary conditions and one initial condition given as ([fig-chp5\fig5.4.pptx](#))

$$\text{IC: } T(x,0) = T_i$$

$$\text{BC: (1) } (\partial T / \partial x)_{@x=0} = 0$$

$$(2) \quad [-k(\partial T / \partial x)_{@x=L}] = h[T(L,t) - T_\infty]$$

The solution will have a functional form of

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h) \text{ (too many variables!)}$$

Non-dimensionalising the dependent variable T will reduce the number of variables as follows:

$$\text{Let } \theta = T - T_\infty, \quad \theta_i = T_i - T_\infty, \quad dT = d\theta$$

$$\text{Define } \theta^* = \theta / \theta_i \quad \text{then, } dT = d\theta = \theta_i d\theta^*$$

A dimensionless spatial coordinate is defined as

$$x^* = x/L \quad \rightarrow \quad dx = L dx^*$$

and dimensionless time

$$t^* = (\alpha t / L^2) = \text{Fo (Fourier No.)}, \quad dt = (L^2 / \alpha) dt^*$$

Substituting the above in the transient conduction equation will give

$$\frac{\theta_i \partial^2 \theta^*}{L^2 \partial x^{*2}} = \frac{1}{\alpha} \frac{\theta_i \partial \theta^*}{(L^2 / \alpha) \partial (t^*)} \quad \text{or} \quad \frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial t^*}$$

And the initial and the boundary conditions become

$$\theta^*(x^*, 0) = 1$$

$$\frac{\partial \theta^*}{\partial x^*} \bigg|_{x^*=0} = 0$$

$$\frac{\partial \theta^*}{\partial x^*} \bigg|_{x^*=1} = -\text{Bi} \theta^*(1, t^*)$$

where $Bi = hL/k$

In dimensionless form, the functional dependence becomes

$$\theta^* = f(x^*, Fo, Bi)$$

much simpler than the representation of function T.

5.5 PLANE WALL WITH CONVECTION

This has an exact solution and also an approximate solution derived from the exact solution.

5.5.1 Exact Solution

No attempt will be made to go through the steps of the exact solution. Referring to [fig-chp5\fig5.6.pptx](#), a plane wall of thickness $2L$ with the assumption that

this thickness is much smaller than the width and height (allows one dimensional justification). For initial wall temperature, $T(x,0) = T_i$ and suddenly immersed in a fluid of T_∞ , the exact solution is in series form and determined as

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*)$$

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)}$$

and the discrete values ζ_n , called eigenvalues are positive roots of the transcendental equation

$$\zeta_n \tan \zeta_n = \text{Bi} \quad \text{or} \quad \tan \zeta_n = \text{Bi} / \zeta_n$$

The first few solutions for $\text{Bi}=10$ are given in the accompanying graph ([fig-chp5\ eigen1.pptx](#)). Roots for different values of Bi are given in the handout.

5.5.2 Approximate Solutions

For values of $\text{Fo} > 0.2$, the infinite series solution can be approximated by the first term of the series only. This will give

$$\theta^* = C_1 \exp(-\zeta_1^2 \text{Fo}) \cos(\zeta_1 x^*)$$

At midplane ($x^*=0$) [fig-chp5\fig5.7.pptx](#)

$$\theta_o^* \equiv \frac{T_o - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo)$$

This will give $\theta^* = \theta_o^* \cos(\zeta_1 x^*)$ [fig-chp5\fig5.8.pptx](#)

The above equation shows that the time dependence of the temperature at any location within the wall is the same as that of the midplane temperature.

5.5.3 Total Energy Transfer

For a time interval from $t=0$ to any time $t>0$, energy conservation equation can be written as

$$E_{in} - E_{out} = \Delta E_{st}$$

For heat transfer from the surface

$$E_{\text{in}} = 0 \quad \text{and this gives} \quad E_{\text{out}} = -\Delta E_{\text{st}} = Q$$

$$\text{Or} \quad Q = -[E(t) - E(0)] = -\int \rho c [T(x,t) - T_i] dV$$

To nondimensionalise, let

$$Q_o = \rho c V (T_i - T_\infty) \quad Q_{\text{max}} \text{ at } t \rightarrow \infty$$

The nondimensional expression will be [fig-](#)

[chp5\fig5.9.pptx](#)

$$\frac{Q}{Q_o} = \int \frac{-[T(x,t) - T_i]}{T_i - T_\infty} \frac{dV}{V} = \frac{1}{V} \int (1 - \theta^*) dV$$

Use the approximate solution and with $V = 2LWH$,

$$dV = (dx)WH = (dx^*L)WH \quad \text{to get } (x^* \rightarrow 0 \text{ to } 1)$$

$$\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^*$$

5.5.4 Additional Considerations

The above solution is applicable to a plane wall, thickness L and insulated on one side ($x^* = 0$).

The foregoing results may be used to determine the transient response of a plane wall to a sudden change in surface temperature $\rightarrow$ equivalent to $h = \infty$ which gives $Bi = \infty$, and T_∞ replaced by T_s .

5.6 RADIAL SYSTEMS WITH CONVECTION

5.6.1 Infinite Cylinder

For an infinite cylinder, the temperature change is in the radial direction only. This approximation can hold true for $L / r_o \geq 10$.

5.6.1 Exact Solution

In dimensionless form, the solution is given in series form as

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*) \quad \text{where} \quad Fo = \frac{\alpha t}{r_o^2} \quad \text{and}$$

$$C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)}$$

And the eigenvalues of ζ_n are positive roots of the transcendental equation

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi$$

[fig-chp5\eigen2.pptx](#)

where J_1 and J_0 are Bessel functions of the first kind of orders one and zero respectively.

5.6.2 Sphere

For a sphere with radius r_o , the exact solution is given by

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \frac{1}{\zeta_n r^*} \sin(\zeta_n r^*) \quad \text{where} \quad Fo = \frac{\alpha t}{r_o^2}$$

$$C_n = \frac{4[\sin(\zeta_n) - \zeta_n \cos(\zeta_n)]}{2\zeta_n - \sin(2\zeta_n)}$$

where the discrete values of ζ_n are the positive roots of the transcendental equation ([fig-chp5\ eigen3.pptx](#))

$$1 - \zeta_n \cot \zeta_n = Bi$$

5.6.2 Approximate Solutions

For the infinite cylinder and sphere the series solution can be approximated by a single term for $Fo > 0.2$ and the time dependence of the temperature at any location is the same as that of the centerline or centerpoint.

Infinite Cylinder

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*) \quad \text{or}$$

$$\theta^* = \theta_o^* J_0(\zeta_1 r^*) \quad \text{where } \theta_o^* \text{ is centerline } T$$

given by ($\theta_r = r^* = 0$)

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo)$$

[fig-chp5\fig5.10.pptx](#) , [fig-chp5\fig5.11.pptx](#)

Sphere

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*) \quad \text{or}$$

$$\theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*) \quad \text{where } \theta_o^* \text{ is centerpoint T}$$

Given by $\theta_o^* = C_1 \exp(-\zeta_1^2 Fo)$

[fig-chp5\fig5.13.pptx](#) , [fig-chp5\fig5.14.pptx](#)

5.6.3 Total Energy Transfer

Using similar procedure as that of the plane wall, the heat transfers can be determined as:

Infinite Cylinders

$$\frac{Q}{Q_0} = 1 - \frac{2\theta_0^*}{\zeta_1} J_1(\zeta_1)$$

[fig-chp5\fig5.12.pptx](#)

Sphere

$$\frac{Q}{Q_0} = 1 - \frac{3\theta_0^*}{\zeta_1^3} [\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)]$$

[fig-chp5\fig5.15.pptx](#)

5.6.4 Additional Considerations

The above also gives the solution when the cylinder or sphere is subjected to a sudden change in surface temperature to T_s . Replace T_∞ by T_s which results due to infinite h value, hence infinite Bi .

Example 5.3

A new process for treatment of a special material is to be evaluated. The material, a sphere of radius $r_o = 5$ mm, is initially in equilibrium at 400°C in a furnace. It is suddenly removed from the furnace and subjected to a two-step cooling process.

Step 1

Cooling in air at 20°C for a period of time t_a until the center temperature reaches a critical value, $T_a(0, t_a) = 335^\circ\text{C}$. For this situation, the convective heat transfer coefficient is $h_a = 10 \text{ W/m}^2\cdot\text{K}$.

After the sphere has reached this critical temperature, the second step is initiated.

Step 2

Cooling in a well-stirred water bath at 20°C, with a convective heat transfer coefficient of $h_w = 6000 \text{ W/m}^2\cdot\text{K}$.

$\rho = 3000 \text{ kg/m}^3$, $k=20 \text{ W/m.K}$, $c=1000 \text{ J/kg.K}$ and
 $\alpha = 6.66 \times 10^{-6} \text{ m}^2/\text{s}$

1. Calculate the time t_a required for step 1 of the cooling process to be completed.
2. Calculate the time t_w required during step 2 of the process for the center of the sphere to cool from 335°C to 50°C .

Solution

1. To check if lumped capacitance method can be used with $L_c = r_o/3$,

$$\text{Bi} = \frac{h_o r_o}{3k} = \frac{10 \times 0.005}{3 \times 20} = 8.33 \times 10^{-4}$$

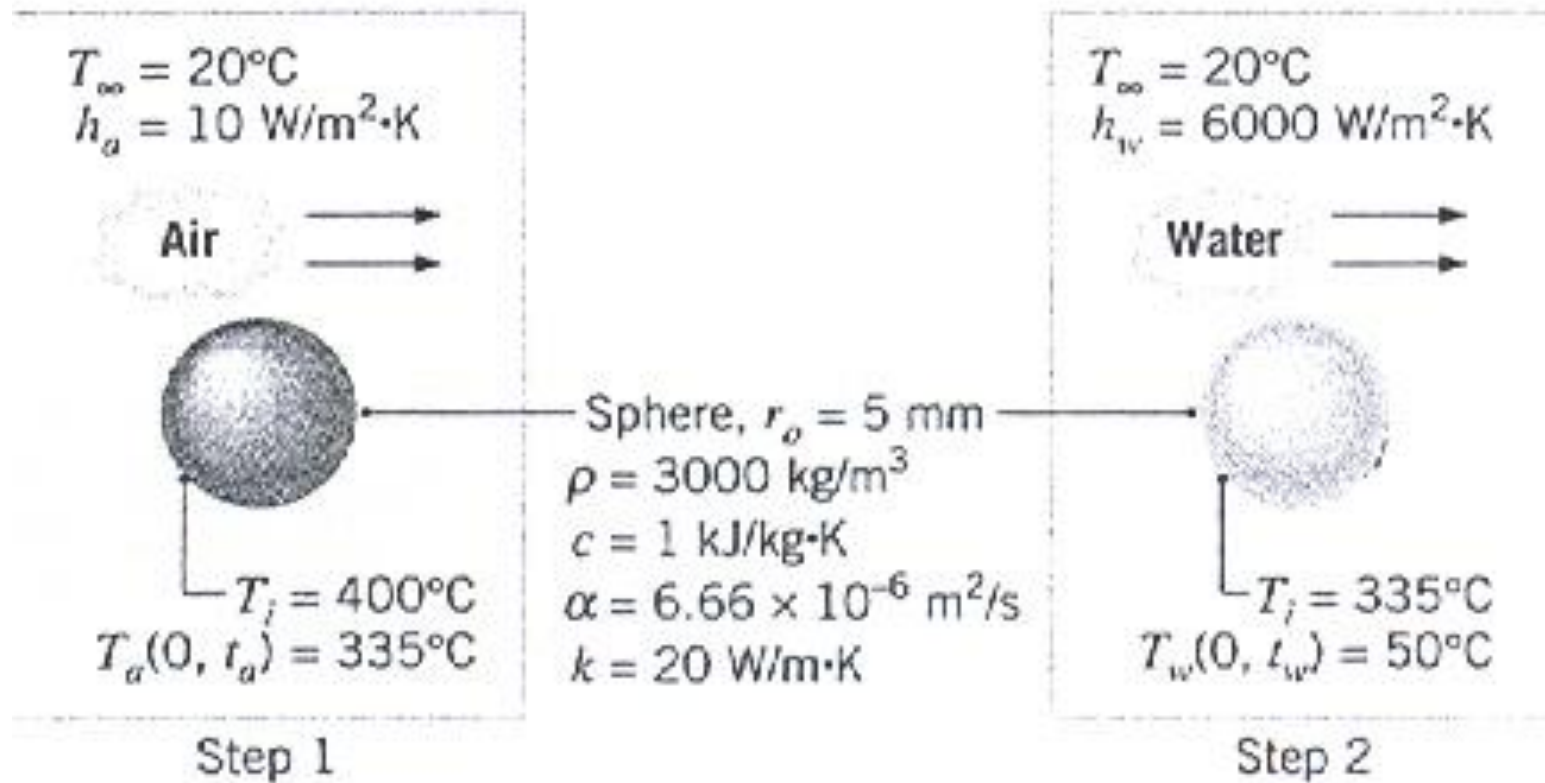


Figure for example 5.3

Indeed the lumped capacitance method can be used.

This will give

$$\begin{aligned} t_a &= \frac{\rho V c}{h_a A_s} \ln \frac{\theta_i}{\theta_o} = \frac{\rho r_o c}{3 h_a} \ln \frac{T_i - T_\infty}{T_a - T_\infty} \\ &= \frac{3000 \times 0.005 \times 1000}{3 \times 20} \ln \frac{400 - 20}{335 - 20} = 94 \text{ s} \end{aligned}$$

2. To check if the lumped capacitance method can be used

$$Bi = \frac{h_o r_o}{3k} = \frac{6000 \times 0.005}{3 \times 20} = 0.50 > 0.1$$

Lumped capacitance method can not be used. A one term approximation can be used.

With the Biot number now defined as

$$Bi = h_w r_o / k = (6000 \times 0.005) / 20 = 1.50$$

The table gives $C_1 = 1.376$ and $\zeta_1 = 1.8$ rad

$$\begin{aligned} Fo &= -\frac{1}{\zeta_1^2} \ln \left[\frac{\theta_o^*}{C_1} \right] = -\frac{1}{\zeta_1^2} \ln \left[\frac{1}{C_1} \times \frac{T(0, t_w) - T_\infty}{T_i - T_\infty} \right] \\ &= -\frac{1}{1.8^2} \ln \left[\frac{1}{1.376} \times \frac{50 - 20}{335 - 20} \right] = 0.82 > 0.2 \end{aligned}$$

One term approximation is justifiable.

$$t_w = Fo \frac{r_o^2}{\alpha} = 0.82 \frac{0.005^2}{6.66 \times 10^{-6}} = 3.1 \text{ s}$$

5.7 THE SEMI-INFINITE SOLID

This is a geometry that is infinite in size in all but one direction having one surface only. The interior is well removed from the surface that it is unaffected by the surface condition.

$$\text{i.e. } T(x \rightarrow \infty, t) = T_i$$

A one dimensional transient equation can be used for such situation. Heat transfer near the surface of the earth or transient response of a thick slab are a few of the examples that can be mentioned.

Three possible situations can exist on the surface as shown in [fig-chp5\fig5.16.pptx](#) .

The familiar equation will be used

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad \text{I.C.} \quad T(x,0) = T_i$$

Interior boundary condition

$$T(x \rightarrow \infty, t) = T_i$$

The above equation can be transformed into an ODE by using a function of the form $\eta = \eta(x,t)$ that will result in $T(\eta)$ instead of $T(x,t)$.

Transformation is done as follows:

Use the transformation equation

$$\eta \equiv x/(4\alpha t)^{1/2}$$

to get the following derivatives

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta} \frac{\partial \eta}{\partial x} = \frac{1}{(4\alpha t)^{1/2}} \frac{dT}{d\eta}$$

$$\begin{aligned} \frac{\partial^2 T}{\partial x^2} &= \frac{d}{d\eta} \left[\frac{\partial T}{\partial x} \right] \frac{\partial \eta}{\partial x} = \frac{d}{d\eta} \left[\frac{1}{(4\alpha t)^{1/2}} \frac{dT}{d\eta} \right] \frac{1}{(4\alpha t)^{1/2}} \\ &= \frac{1}{4\alpha t} \frac{d^2 T}{d\eta^2} \end{aligned}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta} \frac{\partial \eta}{\partial t} = -\frac{x}{2t(4\alpha t)^{1/2}} \frac{dT}{d\eta} = -\frac{\eta}{2t} \frac{dT}{d\eta}$$

Substitution gives

$$\frac{d^2T}{d\eta^2} = -2\eta \frac{dT}{d\eta}$$

Boundary and initial conditions for case (1):

$$\text{For } x = 0 \rightarrow \eta = 0$$

$$T(0,t) \rightarrow T(\eta=0) = T_s$$

For $x \rightarrow \infty$, and $t=0$ (both corresponding to $\eta \rightarrow \infty$)

$$T(\eta \rightarrow \infty) = T_i$$

The equation to be solved is

$$\frac{d\left(\frac{dT}{d\eta}\right)}{d\eta} = -2\eta \frac{dT}{d\eta} \quad \text{or} \quad \frac{d\left(\frac{dT}{d\eta}\right)}{\frac{dT}{d\eta}} = -2\eta d\eta$$

Integration gives

$$\ln\left(\frac{dT}{d\eta}\right) = -\eta^2 + C_1' \quad \text{or} \quad \frac{dT}{d\eta} = C_1 \exp(-\eta^2)$$

Integrating a second time, we obtain

$$T = C_1 \int_0^\eta \exp(-u^2) du + C_2$$

u is a dummy variable.

Applying the boundary condition $T(\eta=0) = T_s$ gives

$$C_2 = T_s \quad \text{The resulting equation will be}$$

$$T = C_1 \int_0^\eta \exp(-u^2) du + T_s$$

Applying the second boundary condition

$T(\eta \rightarrow \infty) = T_i$ results in

$$T_i = C_1 \int_0^{\infty} \exp(-u^2) du + T_s$$

Evaluating the definite integral gives

$$C_1 = \frac{2(T_i - T_s)}{\pi^{1/2}}$$

Hence the temperature distribution may be expressed as

$$\frac{T - T_s}{T_i - T_s} = (2 / \pi^{1/2}) \int_0^{\eta} \exp(-u^2) du \equiv \text{erf } \eta$$

erf η , called Gaussian error function is a standard mathematical function whose values are given in tables.

The surface heat flux may be obtained by using Fourier's law at $x = 0$ as follows:

$$\begin{aligned}
 q_s'' &= -k \frac{\partial T}{\partial x} @_{x=0} = -k(T_i - T_s) \frac{d(\text{erf } \eta)}{d\eta} \frac{d\eta}{dx} @_{\eta=0} \\
 &= -k(T_i - T_s)(2 / \pi^{1/2}) \exp(-\eta^2)(4\alpha t)^{-1/2} @_{\eta=0} \\
 q_s'' &= \frac{k(T_s - T_i)}{(\pi\alpha t)^{1/2}}
 \end{aligned}$$

Analytical solutions can also be determined for Case (2) and Case (3). The summary of the results is given below.

Case (1) Constant Surface Temperature: $T(0,t)=T_s$

$$\frac{T(x,t) - T_s}{T_i - T_s} = \operatorname{erf}(\eta) = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}}$$

Temperature within the medium monotonically approach T_s with increasing t . Surface temperature gradient, and hence the surface heat flux decreases as $t^{-1/2}$.

Case (2) Constant Surface Heat Flux: $q_s'' = q_o''$

$$T(x, t) - T_i = \frac{2q_o''(\alpha t / \pi)^{1/2}}{k} \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_o'' x}{k} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Surface temperature $T(0, t) = T_s(t)$ for a constant heat flux increases monotonically as $t^{1/2}$.

Case (3) Surface Convection: $-k \frac{\partial T}{\partial x} \big|_{x=0} = h[T_\infty - T(0, t)]$

$$\frac{T(x, t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right) \right] \left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right) \right]$$

$\text{erfc } \eta \equiv 1 - \text{erf } \eta$ - Complementary error function

For $h \rightarrow \infty$, the 2nd expression goes to zero, $T_\infty = T_s$.

Gives the same result as case 1.

T_s and temperatures within the medium approach the fluid temperature T_∞ with increasing time. As T_s approaches T_∞ , there will be a reduction in heat flux.

[fig-chp5\fig5.17.pptx](#) shows the temperature distribution on the surface and in the medium.

An interesting application of case 1 is when two semi-infinite solids, at temperatures $T_{A,i}$ and $T_{B,i}$ ($T_A > T_B$) are placed in

contact at their free surfaces as shown in [fig-chp5\fig5.18.pptx](#) . For negligible contact resistance, this will dictate a surface temperature T_s at time of contact on both surfaces.

$$q_{S,A}'' = q_{S,B}'' , \quad -\frac{k_A (T_s - T_{A,i})}{(\pi \alpha_A t)^{1/2}} = -\frac{k_B (T_s - T_{B,i})}{(\pi \alpha_B t)^{1/2}}$$

And solving for T_s gives

$$T_s = \frac{(k\rho c)_A^{1/2} T_{A,i} + (k\rho c)_B^{1/2} T_{B,i}}{(k\rho c)_A^{1/2} + (k\rho c)_B^{1/2}}$$